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Recursive characterization of computable real-valued functions and relations

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Abstract

Corresponding to the definition of μ -recursive functions we introduce a class of recursive relations in metric spaces such that each relation is generated from a class of basic relations by a finite number of applications of some specified operators. We prove that our class of recursive relations essentially coincides with our class of densely computable relations, defined via Turing machines. In the special case of the real numbers our subclass of recursive functions coincides with the classical class of computable real-valued functions, defined via Turing machines by Grzegorzczuk, Lacombe and others.

1. Introduction

1.1. Discrete computability

In the theory of discrete computability there are two main ways to introduce effectivity for functions:¹

(a) A function $f : \subseteq \mathbb{N} \rightarrow \mathbb{N}$ is called *computable* if there is a Turing machine which computes a name of the output from a name of the input by a finite number of elementary symbol manipulations.

(b) A function $f : \subseteq \mathbb{N} \rightarrow \mathbb{N}$ is called *recursive* if it can be generated from a class of basic functions by a finite number of applications of certain operators.

Here a *name* of a natural number n is a word w over a finite alphabet which encodes n (e.g. $w = 10^n 1$).

The class of *basic functions* consists of the projections and of the functions related with the *Peano structure* of \mathbb{N} :

$$0_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{N}, n \mapsto 0, \quad S_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{N}, n \mapsto n + 1.$$

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¹ We use the notation $f : \subseteq X \rightarrow Y$ for *partial functions* with $\text{dom}(f) \subseteq X$ and $\mathbb{N} := \{0, 1, 2, \dots\}$ for the set of *natural numbers*.

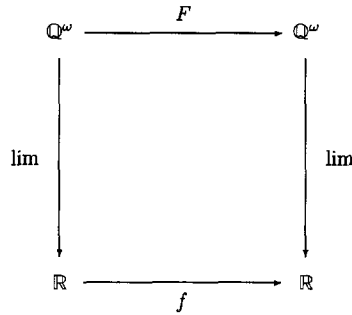


Fig. 1. Computability of real-valued functions.

The *operators* used to generate functions are

substitution, primitive recursion and minimization.

The type (a) approach, which is closer to physical computers, is due to Turing ([34]). The algebraic type (b) approach, which is closer to mathematical thinking, is due to Kleene ([18]) and is based on the work of Herbrand and Gödel.

It is well-known that the class of computable functions coincides with the class of recursive functions. Furthermore, we have:

Church's Thesis. The definitions of computable resp. recursive functions are formalizations of the intuitive notion of effective computability.

1.2. Computability of real-valued functions

In the theory of continuous computability, where functions $f: \subseteq \mathbb{R} \rightarrow \mathbb{R}$ are considered, the situation is quite different. On the one hand, there are the classical Turing machine based type (a) definitions of Grzegorzczuk ([10, 11]) and Lacombe ([23]) which were investigated and generalized by Hauck; Pour-El and Richards, Friedman and Ko, Kreitz and Weihrauch and others. In these approaches an approximation of the output with arbitrary precision is computed from a suitable approximation of the input.²

- (a) A function $f: \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called *computable* if there is a computable operator $F: \subseteq \mathbb{Q}^\omega \rightarrow \mathbb{Q}^\omega$ such that $F(q)$ is a name of $f(x)$, provided that q is a name of x (Fig. 1).

Here a sequence of rational numbers $q = (q_n)_{n \in \mathbb{N}}$ is called a *name* of $x \in \mathbb{R}$ if $(q_n)_{n \in \mathbb{N}}$ is a Cauchy sequence which converges to x fast, i.e.

$$x = \lim_{n \rightarrow \infty} q_n \text{ and } (\forall m > n) |q_n - q_m| < 2^{-n}.$$

² We use the notation $X^\omega := \{(x_n)_{n \in \mathbb{N}} | x_n \in X \text{ for all } n \in \mathbb{N}\}$ for sequence spaces.

Furthermore, an operator $F : \subseteq \mathbb{Q}^\omega \rightarrow \mathbb{Q}^\omega$ is called *computable* if the corresponding functional $F' : \subseteq \mathbb{Q}^\omega \times \mathbb{N} \rightarrow \mathbb{Q}, (q, k) \mapsto F(q)(k)$ is computable with a Turing machine (cf. [10]).

It is easy to realize that approximately computing a function requires the continuity of the function. Otherwise a demanded precision of the output could hardly be computed from an approximation of the input in general. Hence it is a theorem in the type (a) approaches that computable functions are continuous. This result leads to the

Thesis of Recursive Analysis. All physically computable functions are continuous.

In addition to the presented type (a) approach there is a kind of type (b) definition of recursive real functions by Blum et al. [1]. In this approach real numbers are viewed as entities and besides all rational functions the discontinuous tests “=” and “ \leq ” are taken as elementary.

(BSS) A function $f : \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called *BSS recursive* if it can be generated from a class of basic functions (consisting of all rational functions and the sign function) by a finite number of applications of certain operators.

The operators used to generate functions in the BSS approach essentially correspond to the operators used in the theory of discrete computability.

The real RAM model, related to this approach, is the main model of computability in computational geometry (cf. [30]). Unfortunately serious problems, caused by degeneracies which are based on discontinuity, show that this model is unrealistic in the sense that it cannot be implemented on physical computers (cf. [6, 16]).

This observation corresponds to a fact which is well-known in numerical analysis: the test “ $x = 0$ ” is critical!

In that way the Thesis of Recursive Analysis is empirically confirmed and as long as it is not falsified we should keep it.

At this place we want to emphasize the three main disadvantages of the BSS model from our point of view:

- **Unrealizability.** There are discontinuous functions like tests which are BSS recursive but non-computable on physical computers.
- **Incompleteness.** There are analytic functions, like the exponential function and the trigonometric functions, which are computable on physical computers but not BSS recursive.
- **Uncountability.** Since arbitrary rational functions are allowed, the class of BSS recursive functions is uncountable. (Especially there are BSS recursive functions which “solve” undecidable discrete problems.)

All together it is quite unlikely that the BSS recursive functions are a suitable candidate for an extension of Church’s Thesis to continuous spaces. Nevertheless, they may be helpful for other purposes (e.g. for the complexity analysis of numerically stable algorithms, cf. [33]).

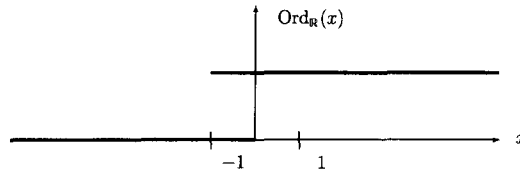


Fig. 2. The relaxed order relation.

1.3. Recursive real-valued relations

In this paper we introduce a class of real-valued recursive relations corresponding to the algebraic type (b) definition:

- (b) A relation $R \subseteq \mathbb{R} \times \mathbb{R}$ is called *recursive* if it can be generated from a class of basic relations by a finite number of applications of certain operators.

Besides the functions of the Peano structure of \mathbb{N} , as defined above, and all projections of finite products of \mathbb{N} and \mathbb{R} , the class of *basic relations* consists of the functions of the *field structure* of \mathbb{R} and of the *relaxed order relation* of \mathbb{R} :

- $0_{\mathbb{R}} : \mathbb{N} \rightarrow \mathbb{R}, n \mapsto 0$, $1_{\mathbb{R}} : \mathbb{N} \rightarrow \mathbb{R}, n \mapsto 1$,
- $\text{Add}_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, (x, y) \mapsto x + y$, $\text{Mul}_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, (x, y) \mapsto x \cdot y$,
- $\text{Neg}_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto -x$, $\text{Inv}_{\mathbb{R}} : \subseteq \mathbb{R} \rightarrow \mathbb{R}, x \mapsto 1/x$.
- $\text{Ord}_{\mathbb{R}} := \{(x, 0) | x < 0\} \cup \{(x, 1) | x + 1 > 0\} \subseteq \mathbb{R} \times \mathbb{N}$.

The *operators*, which will be defined in Section 3, are

juxtaposition, composition, iteration, minimization and limitation.

Here the first four operators are natural generalizations of the usual operators (as used by Blum et al. [1]). The limitation operator is an additional operator which guarantees the completeness of our class of recursive relations. The class of recursive relations is countable since the set of basic relations is. The relaxed order relation is the only basic relation which is not a function (Fig. 2). Fig. 3 compares recursive relations with BSS recursive relations. We will show that there is a reasonable notion of continuity for relations such that all recursive relations are continuous.

Furthermore, we introduce the Turing machine based type (a) notion of *dense computability* which is an easy generalization of the type (a) notion of computability for functions:

- (a) A relation $R \subseteq \mathbb{R} \times \mathbb{R}$ is called *densely computable* if there is a computable operator $F : \subseteq \mathbb{Q}^{\omega} \rightarrow (\mathbb{Q}^{\omega})^{\omega}$ such that $F(q)$ is a name of a sequence $(y_n)_{n \in \mathbb{N}}$ of real numbers which is dense in $R(x)$, provided that q is a name of x .

Our main result states that

the recursive relations are essentially the densely computable ones.

While the type (a) and (b) definitions in the theory of discrete computability were developed at the same time, it is surprising that our characterization seems to be the first complete type (b) characterization of the real-valued computable operations.

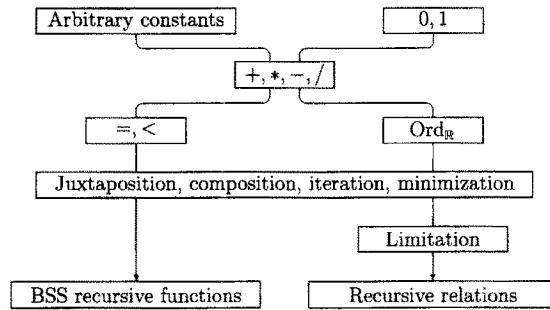


Fig. 3. Models of computability on the real numbers.

Nevertheless, there are some type (b) characterizations of computable functionals (cf. [10, 17, 12]). Furthermore, one can deduce from the effective Weierstraß Theorem (proved independently by Pour-El and Caldwell [28] and Hauck [14]) that computable functions $f : [a, b] \rightarrow \mathbb{R}$ with computable $a, b \in \mathbb{R}$ are “uniformly recursive” in the following sense: they can be generated without using the order relation $\text{Ord}_{\mathbb{R}}$ and without applying the minimization operator to real-valued functions (cf. [14, 32]). Recently there are some other approaches to recursion on the real numbers (cf. [9, 25]).

1.4. Why we do consider relations

Now the question may arise, why we do use relations to characterize computability instead of confining ourselves to functions. The answer is as follows: if we wish to perform computations which are not straightforward but allow branchings depending on the input, we have to gain finite information about a given real number, i.e. we need operations of the type $t : \mathbb{R} \rightarrow \mathbb{N}$. Since continuous functions from a connected space to a discrete space are constant, we know that interesting operations of this type cannot have both properties:

continuity and functionality.

So, asked to choose one of these conflicting properties we have decided to drop functionality in order to keep continuity, which is a necessary condition to meet the Thesis of Recursive Analysis.

The type (a) approaches evade this problem on the level of the names: the sequence space \mathbb{Q}^{ω} , used to represent real numbers, is zero-dimensional and hence totally disconnected, i.e. non-trivial continuous tests are available.

It may be intuitive to think of our relations as “non-deterministic operations”. But the reader should be warned that this notion of non-determinism is not the usual one. Especially this type of non-determinism is compatible with deterministic computers: the result of a computation does not depend on a random choice of the computation path but on the actual representation of the infinite input. A program of this type is correct for a fixed input if *each* corresponding computation path leads to a valid result.

Another reason to extend the investigation to relations is that relations are of interest by their own. For example the relation $\text{ROOTS} \subseteq \mathbb{C}^n \times \mathbb{C}^n$, defined by

$$\text{ROOTS} := \left\{ (a, w) \left| \{w_0, \dots, w_{n-1}\} \text{ is the set of zeros of } z^n + \sum_{k=0}^{n-1} a_k z^k \right. \right\},$$

is computable (cf. [20]) but it has no continuous selector, i.e. there is no continuous function $f: \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that³ $\text{graph}(f) \subseteq \text{ROOTS}$.

1.5. Survey

Section 2 provides continuous relations and some topological tools. Afterwards in Section 3 the recursion operators are defined and explained. Section 4 deals with recursive space systems and recursive relations. Finally in Section 5 to 8 the class of recursive relations is compared with the class of densely computable relations, defined via Turing machines in computable metric spaces. Thereby in Section 6 it is shown that the recursion operators preserve dense computability and in Section 7 it is shown that all strongly densely computable relations can be generated by the recursion operators. Furthermore, a normal form theorem, which corresponds to Kleene's Normal Form Theorem, is deduced and in Section 8 the results are applied to the spaces \mathbb{R} and $\mathcal{C}[0, 1]$.

2. Topological preliminaries

In this section we prepare the topological framework for our definition of recursive relations. We introduce continuous relations as well as two limits and two distances for sets.

2.1. Continuous relations

We start with some basic set theoretic notations for relations. Generalizing the notation $f: \subseteq X \rightarrow Y$ for partial functions we will write $R: \subseteq X \leftrightarrow Y$ for partial relations to emphasize that we will use relations from an *operational* point of view. Hence we are free to identify $X \times Y \times Z := (X \times Y) \times Z$ with $X \times (Y \times Z)$ since our notation $R: \subseteq X \leftrightarrow Y \times Z$ resp. $R: \subseteq X \times Y \leftrightarrow Z$ uniquely characterizes domain and range of R . More precisely a relation⁴ $\mathcal{R}: \subseteq X \leftrightarrow Y$ is a triple (R, X, Y) with $R \subseteq X \times Y$. For simplicity we will not distinguish R from \mathcal{R} .

Let $R: \subseteq X \leftrightarrow Y$ be a relation, $U \subseteq X$, $V \subseteq Y$ and $x \in X, y \in Y$. We define

$$R(x) := \{y \in Y \mid (x, y) \in R\},$$

$$R^{-1}(y) := \{x \in X \mid (x, y) \in R\},$$

³ We use the notation $\text{graph}(f) := \{(x, y) \in X \times Y \mid f(x) = y\}$ for the *graph* of a function $f: \subseteq X \rightarrow Y$.

⁴ Sometimes this kind of operation is called *correspondence* (cf. [2, Section 3.1, Definition 2, p. 96]).

$$R(U) := \{y \in Y \mid R^{-1}(y) \cap U \neq \emptyset\},$$

$$R^{-1}(V) := \{x \in X \mid R(x) \cap V \neq \emptyset\},$$

$$\text{dom}(R) := \{x \in X \mid (\exists y \in Y) (x, y) \in R\},$$

$$\text{range}(R) := \{y \in Y \mid (\exists x \in X) (x, y) \in R\}.$$

Now we introduce the notion of continuity for relations.

Definition 1 (*Continuous relations*). Let X, Y be topological spaces and $R : \subseteq X \leftrightarrow Y$ be a relation. Then R is *continuous* in $(x, y) \in R$ if

$$(\forall \text{ neighbourhoods } V \text{ of } y)(\exists \text{ neighbourhood } U \text{ of } x)(\forall \hat{x} \in U \cap \text{dom}(R)) \\ V \cap R(\hat{x}) \neq \emptyset.$$

Furthermore, R is *continuous* if R is continuous in all points $(x, y) \in R$.

The definition generalizes the notion of continuity of functions. Namely, if $f : \subseteq X \rightarrow Y$ is a function, then obviously

$$f \text{ is continuous} \iff \text{graph}(f) \text{ is continuous.}$$

Continuity for relations has already been studied by other authors (cf. [7, pp. 70–71]). In the most interesting case, where the images of the continuous relations are closed, a remainder of topological functionality is preserved.

Definition 2 (*Relations with closed images*). Let X, Y be topological spaces. Then a relation $R : \subseteq X \leftrightarrow Y$ is said to have *closed images* if $R(x)$ is closed for all $x \in \text{dom}(R)$.

If $\mathcal{A}(Y)$ is the set of all non-empty closed subsets of Y then to each relation $R : \subseteq X \leftrightarrow Y$ with closed images a set-valued function $\hat{R} : \subseteq X \rightarrow \mathcal{A}(Y), x \mapsto R(x)$ is associated. Then R is continuous in our sense if \hat{R} is lower semi-continuous in the sense of Kuratowski ([21, Ch. I, Section 18 I, p. 173]).

We give some characterizations of continuous relations which generalize the corresponding characterizations of continuous functions. The proofs are omitted.

Lemma 3 (Preimage condition for continuity). *Let X, Y be topological spaces and $R : \subseteq X \leftrightarrow Y$. Then R is continuous if and only if for any open set $V \subseteq Y$ the set $R^{-1}(V)$ is open in $\text{dom}(R)$.*

Lemma 4 (Closure condition for continuity). *Let X, Y be topological spaces and $R : \subseteq X \leftrightarrow Y$. Then R is continuous if and only if for any set $A \subseteq \text{dom}(R)$ the inclusion $R(\overline{A}) \subseteq \overline{R(A)}$ holds.⁵*

⁵ We use the notation \overline{A} for the *closure* of a subset $A \subseteq X$ of a topological space X .

2.2. Operations on sets

Since we want to introduce a limit operator which operates on the images of relations we need some operations on sets. We introduce the lower and upper limit of sequences of sets as well as the lower and upper distance for sets, as defined by Hausdorff ([15, Ch. 6, Section 29, pp. 145–150]).⁶

Definition 5 (*Lower and upper limit*). Let (X, d) be a metric space and let $(A_n)_{n \in \mathbb{N}}$ be a sequence of subsets of X .

(1) The *lower limit* of $(A_n)_{n \in \mathbb{N}}$ is defined by $\lim_{n \rightarrow \infty}^< A_n := \{x \in X \mid \lim_{n \rightarrow \infty} d(x, A_n) = 0\}$.

(2) The *upper limit* of $(A_n)_{n \in \mathbb{N}}$ is defined by $\lim_{n \rightarrow \infty}^> A_n := \{x \in X \mid \liminf_{n \rightarrow \infty} d(x, A_n) = 0\}$.

(3) The sequence $(A_n)_{n \in \mathbb{N}}$ is called *convergent* if $A := \lim_{n \rightarrow \infty}^< A_n = \lim_{n \rightarrow \infty}^> A_n$. Then the *limit* of $(A_n)_{n \in \mathbb{N}}$ is defined by $\lim_{n \rightarrow \infty} A_n := A$.

Now we state an easy property of the limit without proof. More details can be found in Hausdorff ([15, Ch. 6, Section 29, pp. 145–150]) and Kuratowski ([21, Ch. II, Section 29, pp. 335–344]).

Lemma 6 (*Closure of the limit*). Let (X, d) be a metric space and let $(A_n)_{n \in \mathbb{N}}$ be a sequence of subsets of X . Then

$$\overline{\lim_{n \rightarrow \infty} A_n} = \lim_{n \rightarrow \infty} \overline{A_n} = \lim_{n \rightarrow \infty} A_n.$$

Corresponding properties hold for the lower and the upper limit.

Now we introduce some special distances for subsets of a metric space which are related to the Hausdorff metric.

Definition 7 (*The lower and the upper distance*). Let (X, d) be a metric space and let $A, B \subseteq X$.

(1) The *lower distance* of A and B is defined by $d^<(A, B) := \sup_{a \in A} d(a, B)$.

(2) The *upper distance* of A and B is defined by $d^>(A, B) := \sup_{b \in B} d(A, b)$.

The next lemma relates the introduced distances with the limits. The proof can be found in Hausdorff ([15, Ch. 6, Section 29, pp. 145–150]).

Lemma 8 (*Distances and limits*). Let (X, d) be a metric space and let $(A_n)_{n \in \mathbb{N}}$ be a sequence of subsets of X . Let $A \subseteq X$ be closed. Then

(1) $\lim_{n \rightarrow \infty} d^<(A_n, A) = 0 \implies \lim_{n \rightarrow \infty}^> A_n \subseteq A$,

(2) $\lim_{n \rightarrow \infty} d^>(A_n, A) = 0 \implies A \subseteq \lim_{n \rightarrow \infty}^< A_n$.

⁶ For metric spaces (X, d_X) we define the *distance* of a point $x \in X$ to a set $A \subseteq X$ by $d_X(x, A) := \inf_{a \in A} d_X(x, a)$.

Finally, we show that a normed Cauchy condition for sequences of sets guarantees the existence of the limit.

Lemma 9 (Normed Cauchy sequences of sets). *Let (X, d) be a metric space and let $(A_n)_{n \in \mathbb{N}}$ be a sequence of non-empty subsets of X , such that*

$$d^<(A_n, A_m) < 2^{-n} \quad \text{for all } m > n.$$

Then $(A_n)_{n \in \mathbb{N}}$ is convergent. Furthermore, if (X, d) is complete then $A := \lim_{n \rightarrow \infty} A_n$ is non-empty and $d^<(A_n, A) \leq 2^{-n}$ for all $n \in \mathbb{N}$.

Proof. We have to show that

$$\lim_{n \rightarrow \infty}^< A_n = \lim_{n \rightarrow \infty}^> A_n.$$

Since “ \subseteq ” holds in general it remains to show “ \supseteq ”.

Hence let $x \in \lim_{n \rightarrow \infty}^> A_n$, i.e. $\liminf_{n \rightarrow \infty} d(x, A_n) = 0$. Then there is a subsequence $(A_{n_k})_{k \in \mathbb{N}}$ of $(A_n)_{n \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} d(x, A_{n_k}) = 0$, where $n_{k+1} > n_k$ for all $k \in \mathbb{N}$. Now let $n \in \mathbb{N}$. Then there is a $k \in \mathbb{N}$ such that $n_k \geq n + 1$ and $d(x, A_{n_k}) < 2^{-n-1}$. Hence for all $m \geq n_k$ we deduce

$$d(x, A_m) \leq d(x, A_{n_k}) + d^<(A_{n_k}, A_m) < 2^{-n},$$

i.e. $\lim_{n \rightarrow \infty} d(x, A_n) = 0$ and $x \in \lim_{n \rightarrow \infty}^< A_n$.

Now assume that (X, d) is complete. Let $n \in \mathbb{N}$ and $x_n \in A_n$. We will show that there is a $x \in A$ for each $k \geq n$ such that

$$d(x_n, x) \leq 2^{-n} + 2^{-k}.$$

Then we can conclude $d^<(A_n, A) \leq 2^{-n}$. Hence fix $k \geq n$ and define $(x_i)_{i \in \mathbb{N}}$ inductively. First choose $x_0 \in A_0, \dots, x_{n-1} \in A_{n-1}, x_{n+1} \in A_{n+1}, \dots, x_k \in A_k$ arbitrarily. Since $d^<(A_i, A_j) < 2^{-i}$ for all $j > i$ there is a $x_{k+1} \in A_{k+1}$ with $d(x_n, x_{k+1}) < 2^{-n}$ and if $x_i \in A_i$ for $i \geq k + 1$ then there is a $x_{i+1} \in A_{i+1}$ with $d(x_i, x_{i+1}) < 2^{-i}$. Obviously, $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence and

$$x := \lim_{i \rightarrow \infty} x_i \in \lim_{i \rightarrow \infty} A_i = A$$

exists, since (X, d) is complete. Furthermore, for all $i \geq k + 1$

$$d(x_n, x_i) \leq d(x_n, x_{k+1}) + \sum_{j=k+1}^{i-1} d(x_j, x_{j+1}) < 2^{-n} + 2^{-k}.$$

Hence, $d(x_n, x) \leq 2^{-n} + 2^{-k}$. \square

3. Recursion operators

In this section we introduce the *recursion operators* which will describe closure properties of recursive relations.

Definition 10 (*Recursion operators*). The following operators are called *recursion operators*:

(1) The operator of *juxtaposition*: for relations $f : \subseteq X \leftrightarrow Y$ and $g : \subseteq X \leftrightarrow Z$ let $(f, g) : \subseteq X \leftrightarrow Y \times Z$ be defined by

$$(f, g)(x) := f(x) \times g(x) = \{(y, z) \in Y \times Z \mid y \in f(x) \text{ and } z \in g(x)\}$$

for all $x \in \text{dom}(f, g) := \text{dom}(f) \cap \text{dom}(g)$.

(2) The operator of *composition*: for relations $f : \subseteq X \leftrightarrow Y$ and $g : \subseteq Y \leftrightarrow Z$ let $g \odot f : \subseteq X \leftrightarrow Z$ be defined by

$$g \odot f(x) := \overline{g(f(x))} = \overline{\{z \in Z \mid (\exists y \in f(x)) z \in g(y)\}}$$

for all $x \in \text{dom}(g \odot f) := \{x \in X \mid f(x) \subseteq \text{dom}(g)\}$.

(3) The operator of *iteration*: for a relation $f : \subseteq X \leftrightarrow X$ let $f^* : \subseteq X \times \mathbb{N} \leftrightarrow X$ be defined by

$$\begin{aligned} f^*(x, 0) &:= \{x\}, \\ f^*(x, n+1) &:= f \odot f^*(x, n) \end{aligned}$$

and abbreviated by $f^n(x) := f^*(x, n)$ for all $x \in X$ and $n \in \mathbb{N}$.

(4) The operator of *minimization*: for a relation $f : \subseteq X \times \mathbb{N} \leftrightarrow Y \times \mathbb{N}$ let $\mu f : \subseteq X \leftrightarrow Y$ be defined by

$$\begin{aligned} \mu f(x) &:= \{y \in Y \mid (\exists n)((y, 0) \in f(x, n) \text{ and} \\ &\quad f(x, 0), \dots, f(x, n-1) \notin Y \times \{0\})\} \end{aligned}$$

for all

$$\begin{aligned} x \in \text{dom}(\mu f) &:= \{x \in X \mid (\exists n)(f(x, n) \subseteq Y \times \{0\} \text{ and} \\ &\quad (x, 0), \dots, (x, n-1) \in \text{dom}(f))\}. \end{aligned}$$

(5) The operator of (*normed*) *limitation*: for a relation $f : \subseteq X \times \mathbb{N} \leftrightarrow Y$ let $\text{Lim} f : \subseteq X \leftrightarrow Y$ be defined by

$$\text{Lim} f(x) := \lim_{n \rightarrow \infty} f(x, n) = \left\{ y \in Y \mid \lim_{n \rightarrow \infty} d(y, f(x, n)) = 0 \right\}$$

for all $x \in \text{dom}(\text{Lim} f) := \{x \in X \mid (\forall m > n) d^<(f(x, n), f(x, m)) < 2^{-n}\}$.

We assume that Z resp. X are topological spaces in (2) resp. (3) and that (Y, d) is a complete metric space in (5). The operators juxtaposition, composition, and iteration are called *primitive recursion operators* and these operators, supplemented by the minimization operator, are called *algebraic recursion operators*.

First, we will make some comments on the definition of the operators.

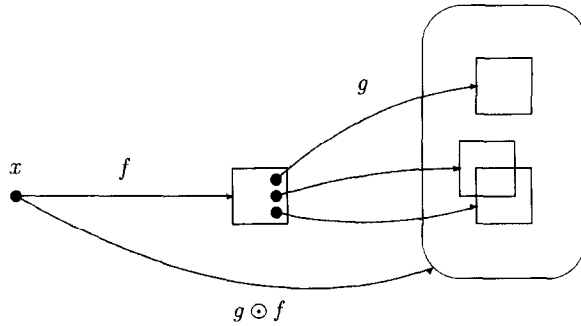


Fig. 4. The composition operator (the oval indicates the closure of the union).

3.1. Juxtaposition

The juxtaposition operator corresponds to the intuition that a relation (f, g) is computable if and only if each component is computable. In the classical theory this operator is not considered explicitly because there are computable bijective tupling functions $\mathbb{N}^2 \rightarrow \mathbb{N}$. For the real numbers there is even no continuous and injective function $\mathbb{R}^2 \rightarrow \mathbb{R}$, such that the juxtaposition becomes useful. Furthermore, we want to allow mixed products in the range of relations. Another advantage is that in the presence of the juxtaposition operator the classical substitution operator can be replaced by the easier composition operator and the primitive recursion operator can be replaced by the easier iteration operator (provided that the projections are available, cf. Lemma 16 and [1, p. 33]).

3.2. Composition

The (demonic) composition has been considered in denotational semantics (cf. [27]) in a similar way. The “all or nothing” condition on the domain guarantees reasonable closure properties, while the pure composition with $\text{dom}(g \circ f) := \{x \in X \mid f(x) \cap \text{dom}(g) \neq \emptyset\}$ does not even preserve continuity in general. If f is a function then we write for short $gf := g \odot f$ (Fig. 4).

3.3. Iteration

The iteration is an easy generalization of the composition.

3.4. Minimization

At first sight our minimization may look a little bit strange. Nevertheless, the definition is a natural generalization of the usual minimization and the idea is very close to programming.

Fig. 5 illustrates the minimization. Assume for the moment that $f : \subseteq X \times \mathbb{N} \leftrightarrow Y \times \mathbb{N}$ is a non-deterministic operation which chooses a $(y, k) \in f(x, n)$. Thereby x is

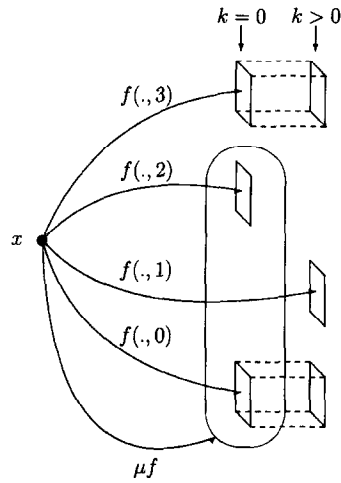


Fig. 5. The minimization operator.

the “input”, n the “loop index”, y is the “result”, and k is the “status” of the result, i.e. $k = 0$ means that the result is valid.

Consider the following program:

```

input  $x$ ;
 $n := 0$ ;
repeat
  choose  $(y, k) \in f(x, n)$ ;
   $n := n + 1$ 
until  $k = 0$ ;
output  $y$ .

```

In this situation $\mu f(x)$ contains all possible results y of the program, in contrast to the classical minimization operator which would yield the corresponding indices n . While in the functional case the results y can be retrieved from the index n , the information contained in the index does not suffice in the relational case. Here some non-deterministic choices of $(y, k) \in f(x, n)$ may yield valid results while others yield invalid results. So, it is necessary to collect the valid results y directly.

The observation that retrieving is impossible in the relational case can be expressed mathematically: for relations $R : \subseteq X \leftrightarrow Y \times Z$ we have $R \subseteq (\text{pr}_1 \odot R, \text{pr}_2 \odot R)$, but in general “ \supseteq ” does not hold, while for a function f instead of R “ $=$ ” holds.

3.5. Limitation

The limitation operator computes the usual limit of the set sequence $(f(x, n))_{n \in \mathbb{N}}$ for each $x \in \text{dom}(\text{Lim} f)$, i.e. $\text{Lim} f(x)$ contains all limits of sequences $(y_n)_{n \in \mathbb{N}}$ with $y_n \in f(x, n)$. The condition $d^<(f(x, n), f(x, m)) < 2^{-n}$ for $m > n$ guarantees that

each $y_n \in f(x, n)$ has a continuation in $f(x, m)$, i.e. there is a $y_m \in f(x, m)$ with $d(y_n, y_m) < 2^{-n}$.

In this case Lemma 9 guarantees that the limit of the sequence $(f(x, n))_{n \in \mathbb{N}}$ exists and that it is a non-empty set for all $x \in \text{dom}(\text{Lim}f)$. Hence Lim is well-defined.

3.6. Properties of the operators

Finally, we state some invariance properties of the operators. The proofs are left to the reader.

Lemma 11 (Invariance properties of the operators). *The juxtaposition, composition, iteration, minimization and limitation operators have the following properties:*

- (a) *Functions are mapped to functions.*
- (b) *Relations with closed images are mapped to relations with closed images.*
- (c) *Continuous relations are mapped to continuous relations.*

We assume that the related spaces are complete metric spaces.

4. Recursive space systems

In this section we define *recursive space systems* and their bases.

Definition 12 (*Recursive space system*). Let X_1, \dots, X_n be complete metric spaces where $X_i = \mathbb{N}$ for at least one $i \in \{1, \dots, n\}$ and let ρ be a set of subrelations of finite products of X_1, \dots, X_n . Then (X_1, \dots, X_n, ρ) is called a *recursive space system* if ρ is closed under application of all recursion operators.

Correspondingly, we can define *primitively* resp. *algebraically recursive space systems* w.r.t. the primitive resp. algebraic recursion operators. Now we introduce the set of *mixed projections* which are rather technical recursive functions.

Definition 13 (*Projections*). Let X_1, \dots, X_n be sets. Then $\pi(X_1, \dots, X_n)$ is defined to be the set of all *mixed projections*

$$\text{pr}_i : Y_1 \times \dots \times Y_k \rightarrow Y_i, (y_1, \dots, y_k) \mapsto y_i$$

where Y_j is a finite product of X_1, \dots, X_n for $j \leq k$.

As usual for structures defined by a closure property, we define the notion of a *base* for a recursive space system.

Definition 14 (*Base of recursive space system*). Let $\mathcal{X} = (X_1, \dots, X_n, \rho)$ be a recursive space system. Then a set β of subrelations of finite products of X_1, \dots, X_n is called a *base* of \mathcal{X} if ρ is the structural closure of $\beta \cup \pi(X_1, \dots, X_n)$ w.r.t. the recursion operators.

Correspondingly, bases for primitively resp. algebraically recursive space systems can be defined. First, we define the most important algebraic recursive space system for the natural numbers.

Definition 15 (*The space \mathbb{N} of natural numbers*). The algebraically recursive space system $(\mathbb{N}, \rho_{\mathbb{N}})$ with the discrete metric on \mathbb{N} is defined by the base $\beta_{\mathbb{N}}$ which consists of the functions of the Peano structure of \mathbb{N}

- $0_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{N}, n \mapsto 0,$
- $S_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{N}, n \mapsto n + 1.$

Now we state that the usual operator of primitive recursion is available in primitively recursive space systems. The proof is left to the reader.

Lemma 16 (Primitive recursion). *Let (X, Y, \mathbb{N}, ρ) be a primitively recursive space system such that $\beta_{\mathbb{N}} \cup \pi(X, Y, \mathbb{N}) \subseteq \rho$ and⁷ all relations in ρ are continuous with closed images. Let $f : \subseteq X \leftrightarrow Y, g : \subseteq Y \times X \times \mathbb{N} \leftrightarrow Y$ be recursive relations, i.e. $f, g \in \rho$. Then $h : \subseteq X \times \mathbb{N} \leftrightarrow Y$, defined by*

$$\begin{aligned} h(x, 0) &:= f(x), \\ h(x, n + 1) &:= g \odot (h, \text{id}_{X \times \mathbb{N}})(x, n), \end{aligned}$$

for all $x \in X, n \in \mathbb{N}$ is recursive, i.e. $h \in \rho$.

Consequently, we get the following corollary.

Corollary 17 (Discrete computability). $\rho_{\mathbb{N}}$ consists of the usual partial computable functions $f : \subseteq \mathbb{N}^n \rightarrow \mathbb{N}^k$ with $n, k \geq 1$.

Now we give some examples of recursive space systems for the most important spaces of analysis. We start with the set of the real numbers.

Definition 18 (*The space \mathbb{R} of real numbers*). The recursive space system $(\mathbb{R}, \mathbb{N}, \rho_{\mathbb{R}})$ with the usual metric is defined by the base $\beta_{\mathbb{R}} \cup \beta_{\mathbb{N}}$, where $\beta_{\mathbb{R}}$ consists of the functions of the field structure of \mathbb{R}

- $0_{\mathbb{R}} : \mathbb{N} \rightarrow \mathbb{R}, n \mapsto 0, 1_{\mathbb{R}} : \mathbb{N} \rightarrow \mathbb{R}, n \mapsto 1,$
 - $\text{Add}_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, (x, y) \mapsto x + y, \text{Mul}_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, (x, y) \mapsto x \cdot y,$
 - $\text{Neg}_{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto -x, \text{Inv}_{\mathbb{R}} : \subseteq \mathbb{R} \rightarrow \mathbb{R}, x \mapsto 1/x$
- and of the *relaxed order relation* of \mathbb{R}
- $\text{Ord}_{\mathbb{R}} := \{(x, 0) \mid x < 0\} \cup \{(x, 1) \mid x + 1 > 0\} \subseteq \mathbb{R} \times \mathbb{N}.$

Sometimes it is useful to have the following generalization of the relaxed order: $L : \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{N} \leftrightarrow \mathbb{N}$, defined by $L(x, y, k) := \text{Ord}_{\mathbb{R}}(2^k \cdot (x - y))$ for all $x, y \in \mathbb{R}$,

⁷ In the case that all relations in ρ are functions, the following condition is superfluous.

$k \in \mathbb{N}$. Obviously, $L \in \rho_{\mathbb{R}}$. More intuitively, we write $x <_k y := L(x, y, k)$ for all $x, y \in \mathbb{R}, k \in \mathbb{N}$, i.e.

$$(x <_k y) \ni \begin{cases} 0 & \iff x < y, \\ 1 & \iff x + 2^{-k} > y. \end{cases}$$

As an example we will show that the square root is a recursive relation. This is an easy application of the Heron algorithm.

Lemma 19 (Square root). *There is a relation $\text{Sqrt}_{\mathbb{R}} : \subseteq \mathbb{R} \leftrightarrow \mathbb{R}$ in $\rho_{\mathbb{R}}$ such that*

$$\text{Sqrt}_{\mathbb{R}}(x) = \{\sqrt{x}\}$$

for all $x \geq 0$.

Proof. Define $f : \subseteq \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$ by

$$\begin{aligned} f(x, 0) &:= 1, \\ f(x, n+1) &:= \frac{1}{2} \left(f(x, n) + \frac{x}{f(x, n)} \right) \end{aligned}$$

for all $x \in \mathbb{R}, n \in \mathbb{N}$. Define $T : \mathbb{R} \times \mathbb{N} \times \mathbb{N} \leftrightarrow \mathbb{R} \times \mathbb{N}$ by

$$T(x, k, n) := f(x, n+2) \times ((f(x, n+1) - f(x, n+2)) <_{k+1} 2^{-k})$$

for all $x \in \mathbb{R}, n, k \in \mathbb{N}$ and define $\text{Sqrt}_{\mathbb{R}} : \subseteq \mathbb{R} \leftrightarrow \mathbb{R}$ by

$$\text{Sqrt}_{\mathbb{R}}(x) := \text{Lim } \mu T(x)$$

for all $x \in \mathbb{R}$. Obviously $f, T, \text{Sqrt}_{\mathbb{R}} \in \rho_{\mathbb{R}}$. We state the well-known a posteriori error estimation for the Heron algorithm:

$$(*) \quad 0 \leq f(x, n+2) - \sqrt{x} \leq f(x, n+1) - f(x, n+2)$$

for all $x \geq 0, n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} f(x, n) = \sqrt{x}$ for all $x \geq 0$.

Let $x \geq 0, k \in \mathbb{N}$. Then there is an $n \in \mathbb{N}$ such that $f(x, n+1) - f(x, n+2) \leq 2^{-k-1}$, i.e. $T(x, k, n) = \{(f(x, n+2), 0)\}$, thus $(x, k) \in \text{dom}(\mu T)$.

Now let $y \in \mu T(x, k)$, i.e. there is an $n \in \mathbb{N}$ such that $y = f(x, n+2)$ and $f(x, n+1) - f(x, n+2) < 2^{-k}$, thus $y - \sqrt{x} \leq 2^{-k}$ by (*). Especially,

$$d^<(\mu T(x, k), \mu T(x, m)) \leq 2^{-k}$$

for all $m > k$ and $x \in \text{dom}(\text{Lim } \mu T)$. Furthermore,

$$\text{Sqrt}_{\mathbb{R}}(x) = \text{Lim } \mu T(x) = \lim_{k \rightarrow \infty} \mu T(x, k) = \{\sqrt{x}\}. \quad \square$$

Other iteration functions with known a posteriori error estimation could be handled correspondingly.

Now, we come to the space of continuous real-valued functions.⁸

Definition 20 (*The space of continuous real-valued functions*). The recursive space system $(\mathcal{C}[0, 1], \mathbb{R}, \mathbb{N}, \rho_{\mathcal{C}[0, 1]})$ with the usual metric, induced by the supremum norm $\| \cdot \|$, is defined by the base $\beta_{\mathcal{C}[0, 1]} \cup \beta_{\mathbb{R}} \cup \beta_{\mathbb{N}}$, where $\beta_{\mathcal{C}[0, 1]}$ consists of the functions of the Banach algebra structure of $\mathcal{C}[0, 1]$, i.e. the functions of the algebraic structure of $\mathcal{C}[0, 1]$

- $1_{\mathcal{C}[0, 1]} : \mathbb{N} \rightarrow \mathcal{C}[0, 1], n \mapsto \hat{1}$,
- $I_{\mathcal{C}[0, 1]} : \mathbb{N} \rightarrow \mathcal{C}[0, 1], n \mapsto \text{id}_{[0, 1]}$,
- $\text{Add}_{\mathcal{C}[0, 1]} : \mathcal{C}[0, 1] \times \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1], (f, g) \mapsto f + g$,
- $\text{SMul}_{\mathcal{C}[0, 1]} : \mathbb{R} \times \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1], (\alpha, f) \mapsto \alpha f$,
- $\text{Mul}_{\mathcal{C}[0, 1]} : \mathcal{C}[0, 1] \times \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1], (f, g) \mapsto f \cdot g$,

and the norm of $\mathcal{C}[0, 1]$

- $\text{Norm}_{\mathcal{C}[0, 1]} : \mathcal{C}[0, 1] \rightarrow \mathbb{R}, f \mapsto \|f\|$.

Here $\hat{c} : [0, 1] \rightarrow \mathbb{R}, x \mapsto c$ denotes the constant function with value c for each $c \in \mathbb{R}$. Obviously, the definition can be generalized to arbitrary real Banach algebras with unit.

The relations in $\beta_{\mathcal{C}[0, 1]}, \beta_{\mathbb{R}}, \beta_{\mathbb{N}}, \pi(\mathcal{C}[0, 1], \mathbb{R}, \mathbb{N})$ are continuous and they have closed images. Hence Lemma 11 yields

Corollary 21 (Continuity of recursive relations). *All recursive relations in $\rho_{\mathbb{R}}, \rho_{\mathcal{C}[0, 1]}$ are continuous and they have closed images.*

5. Recursive and computable relations

In this section we want to compare recursive relations with classically computable functions resp. relations. As a general framework for the classical notion we use the *Type 2 theory* of Kreitz and Weihrauch (cf. [22, 35, 37, 38]), which is a very far developed Turing machine based approach to computability in topological spaces. In Type 2 theory Grzegorzczuk's and Lacombe's original definition of computable real functions is generalized to T_0 -spaces with countable bases. Thereby the computability structure on X is induced by a *representation*, which is a surjective mapping $\delta : \subseteq \mathbb{B} \rightarrow X$, where $\mathbb{B} := \mathbb{N}^{\mathbb{N}}$ is *Baire's space* with the usual product topology. In this situation we call (X, δ) a *represented space*. This concept is used in a similar way by Hauck (cf. [13]). In the real case the definition of Kreitz and Weihrauch coincides with the definition of Ko and Friedman (cf. [19, 20]).

5.1. Computable and densely computable relations

First, we introduce the usual notion of computability for relations (cf. [35, 38]).

⁸ We define $\mathcal{C}[0, 1] := \{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ continuous}\}$ and $\|f\| := \sup_{x \in [0, 1]} |f(x)|$.

Definition 22 (*Computable relations*). Let $(X, \delta_X), (Y, \delta_Y)$ be represented spaces. Then a relation $R : \subseteq X \leftrightarrow Y$ is called (δ_X, δ_Y) -*computable* if there is a computable function $F : \subseteq \mathbb{B} \rightarrow \mathbb{B}$ such that

$$\delta_Y F(p) \in R\delta_X(p) \quad \text{for all } p \in \text{dom}(R\delta_X).$$

Furthermore, R is called *strongly* (δ_X, δ_Y) -*computable* if additionally

$$p \notin \text{dom}(F) \quad \text{for all } p \in \text{dom}(\delta_X) \setminus \text{dom}(R\delta_X).$$

The definition specializes to functions $f : \subseteq X \rightarrow Y$ in the following way:

$$f \text{ is } (\delta_X, \delta_Y)\text{-computable} : \iff \text{graph}(f) \text{ is } (\delta_X, \delta_Y)\text{-computable}.$$

For our purposes this notion of computability is too weak. One reason is that for each name p of x the result $F(p)$ is only a name of an arbitrary element of $R(x)$. Nothing guarantees that the whole image $R(x)$ is covered in any way. Hence each extension R' of a (δ_X, δ_Y) -computable relation R with $\text{dom}(R') = \text{dom}(R)$ is (δ_X, δ_Y) -computable too. Especially, the set of (δ_X, δ_Y) -computable relations with fixed non-empty domain is not countable, presupposed that Y is not countable. Hence, the class of (δ_X, δ_Y) -computable relations can not coincide precisely with the countable class of recursive relations, already by a cardinality argument. Therefore, we introduce the notion of a *densely computable relation*, which has to be defined in the context of topological spaces. We use the following technical notation: for each function $F : \subseteq \mathbb{B} \rightarrow \mathbb{B}$ define $F_n : \subseteq \mathbb{B} \rightarrow \mathbb{B}$ by⁹

$$F_n(p)(k) := F(p)(n, k)$$

for all $p \in \mathbb{B}$ and $n, k \in \mathbb{N}$, i.e. $\text{dom}(F_n) = \text{dom}(F)$ for all $n \in \mathbb{N}$.

Definition 23 (*Dense computability of relations*). Let $(X, \delta_X), (Y, \delta_Y)$ be represented topological spaces. Then a relation $R : \subseteq X \leftrightarrow Y$ is called *densely* (δ_X, δ_Y) -*computable* if there is a computable function $F : \subseteq \mathbb{B} \rightarrow \mathbb{B}$ such that

$$\bigcup_{n=0}^{\infty} \{\delta_Y F_n(p)\} = R\delta_X(p) \quad \text{for all } p \in \text{dom}(R\delta_X).$$

Furthermore, R is called *strongly densely* (δ_X, δ_Y) -*computable* if additionally

$$p \notin \text{dom}(F) \quad \text{for all } p \in \text{dom}(\delta_X) \setminus \text{dom}(R\delta_X).$$

This definition specializes to functions correspondingly to the definition of computable relations.

⁹ We use the notation $\langle \rangle : \mathbb{N}^2 \rightarrow \mathbb{N}, (n, k) \mapsto \langle n, k \rangle := \frac{1}{2}(n+k)(n+k+1) + k$ for Cantor's bijective and computable tupling function.

Now for each name p of x the result $F(p)$ is a name of a sequence $(y_n)_{n \in \mathbb{N}}$ which is dense in $R(x)$. It is very easy to see that dense computability really is a stronger notion than computability.

Lemma 24 (Dense computability implies computability). *Let $(X, \delta_X), (Y, \delta_Y)$ be represented topological spaces. Then for $R : \subseteq X \leftrightarrow Y$ the following holds:*

$$\begin{aligned} R \text{ (strongly) densely } (\delta_X, \delta_Y)\text{-computable} \\ \implies R \text{ (strongly) } (\delta_X, \delta_Y)\text{-computable.} \end{aligned}$$

Furthermore, it is obvious that computability and dense computability coincide as far as only functions are considered.

Lemma 25 (Dense computability and computability for functions). *Let $(X, \delta_X), (Y, \delta_Y)$ be represented topological spaces. Then for $f : \subseteq X \rightarrow Y$ the following statements are equivalent:*

- (1) f is (strongly) densely (δ_X, δ_Y) -computable,
- (2) f is (strongly) (δ_X, δ_Y) -computable.

Now we want to show that dense computability is a reasonable computability notion, i.e. equivalent representations induce the same kind of computability. Here for two representations $\delta, \delta' : \subseteq \mathbb{B} \rightarrow X$ computable reducibility is defined by

$$\delta \leq_c \delta' : \iff (\exists F : \subseteq \mathbb{B} \rightarrow \mathbb{B} \text{ computable})(\forall p \in \text{dom}(\delta)) \delta(p) = \delta'(F(p)),$$

and \equiv_c is the induced equivalence.

Lemma 26 (Invariance of dense computability). *Let $(X, \delta_X), (Y, \delta_Y)$ be represented topological spaces and let δ'_X, δ'_Y be representations such that $\delta'_X \equiv_c \delta_X$ and $\delta'_Y \equiv_c \delta_Y$. Then for $R : \subseteq X \leftrightarrow Y$ the following statements are equivalent:*

- (1) R is (strongly) densely (δ_X, δ_Y) -computable,
- (2) R is (strongly) densely (δ'_X, δ'_Y) -computable.

The proofs are left to the reader.

5.2. Computable metric spaces

Since all results will be formulated in the context of metric spaces we need computable metric spaces (cf. [36, 37]).

Definition 27 (Computable metric space). *Let (X, d) be a separable metric space with a total numbering $\alpha : \mathbb{N} \rightarrow D$ of a dense subset $D \subseteq X$. Define*

$$D_{<} := \{(n, k, q) \in \mathbb{N} \times \mathbb{N} \times \mathbb{Q} \mid d(\alpha(n), \alpha(k)) < q\}$$

and $D_{>}$ correspondingly with “ $>$ ” instead of “ $<$ ”.

- (1) (X, d, D, α) is called *densely enumerated metric space*.
- (2) (X, d, D, α) is called *semi-computable metric space* if $D_<$ is recursively enumerable.
- (3) (X, d, D, α) is called *computable metric space* if $D_<$ and $D_>$ are recursively enumerable.

In densely enumerated metric spaces we can represent elements by Cauchy sequences.

Definition 28 (*Cauchy representation*). Let (X, d, D, α) be a densely enumerated metric space. Then the *Cauchy representation* of X is defined by

$$\delta_X : \subseteq \mathbb{B} \rightarrow X, p \mapsto \lim_{n \rightarrow \infty} \alpha p(n),$$

where $\text{dom}(\delta_X) := \{p \in \mathbb{B} \mid (\forall m > n) d(\alpha p(n), \alpha p(m)) < 2^{-n} \text{ and } \lim_{n \rightarrow \infty} \alpha p(n) \in X\}$.

We call a representation $\delta : \subseteq \mathbb{B} \rightarrow X$ of a densely enumerated metric space (X, d, D, α) *computably admissible* if it is computably equivalent to the Cauchy representation δ_X . Now we define a second representation of densely enumerated metric spaces with interesting properties.

Definition 29 (*Relaxed representation*). Let (X, d, D, α) be a densely enumerated metric space. Then the *relaxed representation* $\delta_X : \subseteq \mathbb{B} \rightarrow X$ of X is defined by

$$\delta_X(p) = x : \iff (\forall n, k) \begin{cases} p\langle n, k \rangle = 0 \implies d(x, \alpha(n)) < 2^{-k}, \\ p\langle n, k \rangle = 1 \implies d(x, \alpha(n)) > 2^{-k-1} \end{cases}$$

for all $x \in X$ and $p \in \{0, 1\}^{\mathbb{N}}$ and $p \notin \text{dom}(\delta_X)$ for $p \in \mathbb{B} \setminus \{0, 1\}^{\mathbb{N}}$.

The following lemma was proved by Matthias Schröder who introduced the relaxed representation (cf. [31]).

Lemma 30 (Relaxed representation). *If (X, d, D, α) is a computable metric space then the relaxed representation $\delta_X : \subseteq \mathbb{B} \rightarrow X$ of X is open, continuous, computably admissible and has compact fibers.*

6. Recursive relations are densely computable

Now we prove that our operators map densely computable relations to densely computable relations. The main idea of the proof is the fact that the class of computable functions in Baire's space is closed under the corresponding operators.

Since we have to consider product spaces we use the *product representation*. If $\delta_X : \subseteq \mathbb{B} \rightarrow X, \delta_Y : \subseteq \mathbb{B} \rightarrow Y$ are representations then the *product representation* $[\delta_X, \delta_Y]$:

$\subseteq \mathbb{B} \rightarrow X \times Y$, resp. $[\delta_X, \text{id}_{\mathbb{N}}]: \subseteq \mathbb{B} \rightarrow X \times \mathbb{N}$ is defined by

$$[\delta_X, \delta_Y](p, q) := (\delta_X(p), \delta_Y(q)), \text{ resp. } [\delta_X, \text{id}_{\mathbb{N}}](p, n) := (\delta_X(p), n),$$

where

$$\langle p, q \rangle(k) := \begin{cases} p(n) & \text{if } k = 2n, \\ q(n) & \text{if } k = 2n + 1, \end{cases}$$

$$\langle p, n \rangle(k) := \begin{cases} n & \text{if } k = 0, \\ p(k - 1) & \text{if } k > 0 \end{cases}$$

for all $p, q \in \mathbb{B}, n, k \in \mathbb{N}$.

Theorem 31 (Densely computable relations and the operators). *Let X, Y, Z be topological spaces with arbitrary representations $\delta_X, \delta_Y, \delta_Z$. Then the following holds:*

(1) *If $f: \subseteq X \leftrightarrow Y, g: \subseteq X \leftrightarrow Z$ are densely (δ_X, δ_Y) - resp. (δ_X, δ_Z) -computable then $(f, g): \subseteq X \leftrightarrow Y \times Z$ is densely $(\delta_X, [\delta_Y, \delta_Z])$ -computable.*

(2) *If $f: \subseteq X \leftrightarrow Y, g: \subseteq Y \leftrightarrow Z$ are densely (δ_X, δ_Y) - resp. (δ_Y, δ_Z) -computable and g is continuous then $g \odot f: \subseteq X \leftrightarrow Z$ is densely (δ_X, δ_Z) -computable.*

(3) *If $f: \subseteq X \leftrightarrow X$ is densely (δ_X, δ_X) -computable and continuous then $f^*: \subseteq X \times \mathbb{N} \leftrightarrow X$ is densely $([\delta_X, \text{id}_{\mathbb{N}}], \delta_X)$ -computable. Here X is assumed to be a T_1 -space.*

(4) *If $f: \subseteq X \times \mathbb{N} \leftrightarrow Y \times \mathbb{N}$ is densely $([\delta_X, \text{id}_{\mathbb{N}}], [\delta_Y, \text{id}_{\mathbb{N}}])$ -computable then $\mu f: \subseteq X \leftrightarrow Y$ is densely (δ_X, δ_Y) -computable.*

Now let (Y, d, D, α) be a semi-computable complete metric space and let δ_Y be a computably admissible representation. Then the following holds:

(5) *If $f: \subseteq X \times \mathbb{N} \leftrightarrow Y$ is densely $([\delta_X, \text{id}_{\mathbb{N}}], \delta_Y)$ -computable then $\text{Lim} f: \subseteq X \leftrightarrow Y$ is densely (δ_X, δ_Y) -computable.*

Proof. (1) Let $f: \subseteq X \leftrightarrow Y, g: \subseteq X \leftrightarrow Z$ be densely (δ_X, δ_Y) - resp. (δ_X, δ_Z) -computable via computable functions $F, G: \subseteq \mathbb{B} \rightarrow \mathbb{B}$. Define $H: \subseteq \mathbb{B} \rightarrow \mathbb{B}$ by

$$H_{\langle i, j \rangle}(p) := \langle F_i(p), G_j(p) \rangle$$

for all $p \in \mathbb{B}, i, j \in \mathbb{N}$. Then H is computable and we have

$$\begin{aligned} \overline{\bigcup_{\langle i, j \rangle=0}^{\infty} \{[\delta_Y, \delta_Z]H_{\langle i, j \rangle}(p)\}} &= \overline{\bigcup_{\langle i, j \rangle=0}^{\infty} \{(\delta_Y F_i(p), \delta_Z G_j(p))\}} \\ &= \overline{\bigcup_{i=0}^{\infty} \{\delta_Y F_i(p)\}} \times \overline{\bigcup_{j=0}^{\infty} \{\delta_Z G_j(p)\}} \\ &= f\delta_X(p) \times g\delta_X(p) \\ &= (f, g)\delta_X(p) \end{aligned}$$

for all $p \in \text{dom}((f, g)\delta_X)$, i.e. (f, g) is densely $(\delta_X, [\delta_Y, \delta_Z])$ -computable via H .

(2) Let $f : \subseteq X \leftrightarrow Y, g : \subseteq Y \leftrightarrow Z$ be densely (δ_X, δ_Y) - resp. (δ_Y, δ_Z) -computable via computable functions $F, G : \subseteq \mathbb{B} \rightarrow \mathbb{B}$. Furthermore, let g be continuous. Define $H : \subseteq \mathbb{B} \rightarrow \mathbb{B}$ by

$$H_{\langle i, j \rangle}(p) := G_j F_i(p)$$

for all $p \in \mathbb{B}, i, j \in \mathbb{N}$. Then H is computable. Let $p \in \text{dom}(g \odot f \delta_X)$. Then $A := \bigcup_{i=0}^{\infty} \{\delta_Y F_i(p)\}$ is dense in $f \delta_X(p)$. Hence by continuity of g and Lemma 4 $g(\overline{A}) \subseteq \overline{g(A)}$. We conclude,

$$\begin{aligned} g \odot f \delta_X(p) &= \overline{g(f \delta_X(p))} = \overline{g(\overline{A})} = \overline{g(A)} \\ &= \overline{\bigcup_{i=0}^{\infty} g \delta_Y F_i(p)} = \overline{\bigcup_{i=0}^{\infty} \bigcup_{j=0}^{\infty} \{\delta_Z G_j F_i(p)\}} \\ &= \overline{\bigcup_{\langle i, j \rangle=0}^{\infty} \{\delta_Z H_{\langle i, j \rangle}(p)\}}. \end{aligned}$$

Hence $g \odot f$ is densely (δ_X, δ_Z) -computable via H .

(3) Let $f : \subseteq X \leftrightarrow X$ be densely (δ_X, δ_X) -computable via a computable function $F : \subseteq \mathbb{B} \rightarrow \mathbb{B}$. Furthermore, let f be continuous. Define $G : \subseteq \mathbb{B} \rightarrow \mathbb{B}$ by

$$G_{\langle i, j \rangle}(p, 0) := p,$$

$$G_{\langle i, j \rangle}(p, n+1) := F_j G_i(p, n)$$

for all $p \in \mathbb{B}, i, j, n \in \mathbb{N}$. Then G is computable. Since X is a T_1 -space one can show that

$$\overline{\bigcup_{\langle i, j \rangle=0}^{\infty} \{\delta_X G_{\langle i, j \rangle}(p, n)\}} = f^*(\delta_X(p), n)$$

for all $(p, n) \in \text{dom}(f^*[\delta_X, \text{id}_{\mathbb{N}}])$ by induction on n . Then f^* is densely $([\delta_X, \text{id}_{\mathbb{N}}], \delta_X)$ -computable via G .

(4) Let $f : \subseteq X \times \mathbb{N} \leftrightarrow Y \times \mathbb{N}$ be densely $([\delta_X, \text{id}_{\mathbb{N}}], [\delta_Y, \text{id}_{\mathbb{N}}])$ -computable via a computable function $F : \subseteq \mathbb{B} \rightarrow \mathbb{B}$. Define $G : \subseteq \mathbb{B} \times \mathbb{N} \rightarrow \mathbb{B} \times \mathbb{N}$ by

$$G(\langle i, p \rangle, n) := \pi F_{\pi_1 \pi_2^a(i)} \langle p, n \rangle$$

for all $p \in \mathbb{B}, i, n \in \mathbb{N}$, where $\pi : \mathbb{B} \rightarrow \mathbb{B} \times \mathbb{N}, \langle p, n \rangle \mapsto (p, n)$ and $\pi_i : \mathbb{N} \rightarrow \mathbb{N}, \langle n_1, n_2 \rangle \mapsto n_i$ for $i \in \{1, 2\}$ are computable projections. Then G and $H : \subseteq \mathbb{B} \rightarrow \mathbb{B}$, defined by

$$H_i(p) := \mu G \langle i, p \rangle$$

for all $p \in \mathbb{B}, i \in \mathbb{N}$, are computable. We want to show

$$\overline{\bigcup_{i=0}^{\infty} \{\delta_Y H_i(p)\}} = \mu f \delta_X(p)$$

for all $p \in \text{dom}(\mu f \delta_X)$. Then $\mu f : \subseteq X \leftrightarrow Y$ is densely (δ_X, δ_Y) -computable via H . Hence let $p \in \text{dom}(\mu f \delta_X)$ and $x := \delta_X(p)$.

“ \subseteq ” Let $y \in \bigcup_{i=0}^{\infty} \{\delta_Y H_i(p)\}$. Since $\mu f(x)$ is closed, it suffices to prove $y \in \mu f(x)$. First there are $q \in \mathbb{B}, i \in \mathbb{N}$ such that $H_i(p) = q$, $\delta_Y(q) = y$. Hence there is a $n \in \mathbb{N}$ such that

$$G(\langle i, p \rangle, n) = (q, 0) \quad \text{and} \quad (\forall k < n) G(\langle i, p \rangle, k) \notin \mathbb{B} \times \{0\}.$$

Furthermore, there are $i_0, \dots, i_{n+1} \in \mathbb{N}$ such that

$$i = \langle i_0, \langle i_1, \langle i_2, \dots, \langle i_{n-1}, \langle i_n, i_{n+1} \rangle \rangle \dots \rangle \rangle \rangle,$$

i.e. $i_n := \pi_1 \pi_2^n(i)$ and

$$F_{i_n} \langle p, n \rangle = \langle q, 0 \rangle \quad \text{and} \quad (\forall k < n) F_{i_k} \langle p, k \rangle \notin \langle \mathbb{B}, \{0\} \rangle.$$

It follows that

$$(y, 0) = (\delta_Y(q), 0) = [\delta_Y, \text{id}_{\mathbb{N}}] F_{i_n} \langle p, n \rangle \in f(\delta_X(p), n) \text{ and} \\ (\forall k < n) [\delta_Y, \text{id}_{\mathbb{N}}] F_{i_k} \langle p, k \rangle \in f(\delta_X(p), k) \setminus (Y \times \{0\}),$$

hence $y \in \mu f \delta_X(p)$.

“ \supseteq ” Let $y \in \mu f(x)$. Then there is an $n \in \mathbb{N}$ such that

$$(y, 0) \in f(x, n) \text{ and } (\forall k < n) f(x, k) \not\subseteq Y \times \{0\}.$$

Hence by assumption there is a sequences $(i_{n,m})_{m \in \mathbb{N}}$ in \mathbb{N} such that

$$(y, 0) = \lim_{m \rightarrow \infty} [\delta_Y, \text{id}_{\mathbb{N}}] F_{i_{n,m}} \langle p, n \rangle \in f(x, k)$$

and for all $m \in \mathbb{N}$

$$\text{pr}_2 \pi F_{i_{n,m}} \langle p, n \rangle = 0.$$

Furthermore, there are $i_0, \dots, i_{n-1} \in \mathbb{N}$ such that

$$(\forall k < n) \text{pr}_2 \pi F_{i_k} \langle p, k \rangle \neq 0.$$

Define,

$$j_m := \langle i_0, \langle i_1, \langle i_2, \dots, \langle i_{n-1}, \langle i_{n,m}, 0 \rangle \rangle \dots \rangle \rangle \rangle,$$

i.e. $i_{n,m} = \pi_1 \pi_2^n(j_m)$ and $(\forall k < n) i_k = \pi_1 \pi_2^k(j_m)$ for all $m \in \mathbb{N}$. Then for all $m \in \mathbb{N}$

$$G(\langle j_m, p \rangle, n) = (\text{pr}_1 \pi F_{i_{n,m}} \langle p, n \rangle, 0) \text{ and} \\ (\forall k < n) G(\langle j_m, p \rangle, k) = (\text{pr}_1 \pi F_{i_k} \langle p, k \rangle, \text{pr}_2 \pi F_{i_k} \langle p, k \rangle) \notin \mathbb{B} \times \{0\}.$$

Therefore,

$$H_{j_m}(p) = \mu G \langle j_m, p \rangle = \text{pr}_1 \pi F_{i_{n,m}} \langle p, n \rangle$$

and

$$\lim_{m \rightarrow \infty} \delta_Y H_{j_m}(p) = \text{pr}_1(\lim_{m \rightarrow \infty} [\delta_Y, \text{id}_{\mathbb{N}}] F_{i_n, m} \langle p, n \rangle) = y,$$

i.e. $y \in \overline{\bigcup_{i=0}^{\infty} \{\delta_Y H_i(p)\}}$.

(5) W.l.o.g. we can assume that δ_Y is the Cauchy representation of Y . Let $f : \subseteq X \times \mathbb{N} \leftrightarrow Y$ be densely $([\delta_X, \text{id}_{\mathbb{N}}], \delta_Y)$ -computable via a computable function $F : \subseteq \mathbb{B} \rightarrow \mathbb{B}$. Define $G : \subseteq \mathbb{B} \rightarrow \mathbb{B}$ by

$$G_{\langle i, k \rangle}(p)(n) := F_{i_n} \langle p, k + n + 3 \rangle (k + n + 3),$$

where

- (a) $i_0 := i$,
- (b) $i_{n+1} \in \mathbb{N}$ is chosen such that

$$d(\alpha(F_{i_{n+1}} \langle p, k + n + 4 \rangle (k + n + 4)), \alpha(G_{\langle i, k \rangle}(p)(n))) < 2^{-n-k-1}$$

for all $p \in \mathbb{B}, n, k, i \in \mathbb{N}$. Since Y is semi-computable, the choice can be made computable and hence G is computable. Now let $p \in \text{dom}(\text{Lim} f \delta_X)$ and $x := \delta_X(p)$. First, we want to show that $G_{\langle i, k \rangle}(p) \in \text{dom}(\delta_Y)$ for all $i, k \in \mathbb{N}$. Hence let $i, k \in \mathbb{N}$. By induction we prove

$G_{\langle i, k \rangle}(p)(n)$ is defined for all $n \in \mathbb{N}$.

$n=0$: Obviously, $G_{\langle i, k \rangle}(p)(0) = F_i \langle p, k + 3 \rangle (k + 3)$ is defined since $p \in \text{dom}(\text{Lim} f \delta_X)$.

$n \rightarrow n + 1$: Now assume that $G_{\langle i, k \rangle}(p)(n)$ is defined. Then i_n exists and

$$y := \delta_Y F_{i_n} \langle p, k + n + 3 \rangle \in f(x, k + n + 3).$$

Since $f(x, k + n + 4)$ is closed there is a $z \in f(x, k + n + 4)$ such that

$$d(y, z) = d(y, f(x, k + n + 4)) \leq d^<(f(x, k + n + 3), f(x, k + n + 4)) < 2^{-k-n-3}.$$

Furthermore, there is a $j \in \mathbb{N}$ such that

$$d(\delta_Y F_j \langle p, k + n + 4 \rangle, z) < 2^{-k-n-3},$$

i.e.

$$\begin{aligned} & d(\alpha(F_j \langle p, k + n + 4 \rangle (k + n + 4)), \alpha(G_{\langle i, k \rangle}(p)(n))) \\ & \leq d(\alpha(F_j \langle p, k + n + 4 \rangle (k + n + 4)), \delta_Y F_j \langle p, k + n + 4 \rangle) \\ & \quad + d(\delta_Y F_j \langle p, k + n + 4 \rangle, z) + d(z, y) \\ & \quad + d(\delta_Y F_{i_n} \langle p, k + n + 3 \rangle, \alpha(F_{i_n} \langle p, k + n + 3 \rangle (k + n + 3))) \\ & < 2^{-k-n-4} + 2^{-k-n-3} + 2^{-k-n-3} + 2^{-k-n-3} \\ & < 4 \cdot 2^{-k-n-3} = 2^{-k-n-1}. \end{aligned}$$

Hence, i_{n+1} , as defined above, exists and $G_{\langle i, k \rangle}(p)(n + 1)$ is defined.

Furthermore, for $m, n \in \mathbb{N}$

$$\begin{aligned}
 (*) \quad & d(\alpha(G_{\langle i, k \rangle}(p)(n)), \alpha(G_{\langle i, k \rangle}(p)(m+n))) \\
 & \leq \sum_{j=0}^{m-1} d(\alpha(G_{\langle i, k \rangle}(p)(j+n)), \alpha(G_{\langle i, k \rangle}(p)(1+j+n))) \\
 & < \sum_{j=0}^{m-1} 2^{-j-n-k-1} < 2^{-n-k} \leq 2^{-n},
 \end{aligned}$$

i.e. $G_{\langle i, k \rangle}(p) \in \text{dom}(\delta_Y)$ for all $i, k \in \mathbb{N}$.

Now we want to show that

$$\overline{\bigcup_{\langle i, k \rangle=0}^{\infty} \{\delta_Y G_{\langle i, k \rangle}(p)\}} = \text{Lim} f \delta_X(p)$$

for all $p \in \text{dom}(\text{Lim} f \delta_X)$. Then $\text{Lim} f$ is densely (δ_X, δ_Y) -computable via G . Hence let $p \in \text{dom}(\text{Lim} f \delta_X)$ and $x := \delta_X(p)$.

“ \subseteq ” Let $y \in \bigcup_{\langle i, k \rangle=0}^{\infty} \{\delta_Y G_{\langle i, k \rangle}(p)\}$. Since $\text{Lim} f(x)$ is closed, it suffices to prove $y \in \text{Lim} f(x)$. First there are $q \in \mathbb{B}, i, k \in \mathbb{N}$ such that

$$G_{\langle i, k \rangle}(p) = q, \quad \delta_Y(q) = y.$$

By assumption,

$$y_n := \delta_Y F_{i_n} \langle p, k+n+3 \rangle \in f(x, k+n+3)$$

and

$$\begin{aligned}
 d(y, y_n) & \leq d(\delta_Y G_{\langle i, k \rangle}(p), \alpha(G_{\langle i, k \rangle}(p)(n))) \\
 & \quad + d(\alpha(F_{i_n} \langle p, k+n+3 \rangle(k+n+3)), \delta_Y F_{i_n} \langle p, k+n+3 \rangle) \\
 & \leq 2^{-n} + 2^{-k-n-3} < 2^{-n+1}
 \end{aligned}$$

for all $n \in \mathbb{N}$. Hence,

$$y = \lim_{n \rightarrow \infty} y_n \in \lim_{n \rightarrow \infty}^< f(x, k+n+3) = \text{Lim} f(x, n).$$

“ \supseteq ” Let $y \in \text{Lim} f(x) = \lim_{n \rightarrow \infty}^< f(x, n)$. Then there is a $k_n \geq n$ and a $y_n \in f(x, k_n+3)$ such that $d(y, y_n) < 2^{-n}$ for each $n \in \mathbb{N}$. There is a sequence $(j_{n,m})_{m \in \mathbb{N}}$ in \mathbb{N} for each $n \in \mathbb{N}$ such that

$$d(\delta_Y F_{j_{n,m}} \langle p, k_n+3 \rangle, y_n) < 2^{-m}$$

for each $n, m \in \mathbb{N}$. Let $j_m := j_{m,m}$ and $y'_m := \delta_Y G_{\langle j_m, k_m \rangle}(p)$ for each $m \in \mathbb{N}$. By (*) especially

$$d(\alpha(G_{\langle j_m, k_m \rangle}(p)(0)), \delta_Y G_{\langle j_m, k_m \rangle}(p)) \leq 2^{-k_m}.$$

We conclude,

$$\begin{aligned}
 d(y, y'_m) &\leq d(y, y_m) + d(y_m, \delta_Y F_{j_m, m} \langle p, k_m + 3 \rangle) \\
 &\quad + d(\delta_Y F_{j_m, m} \langle p, k_m + 3 \rangle, \alpha(F_{j_m, m} \langle p, k_m + 3 \rangle (k_m + 3))) \\
 &\quad + d(\alpha(G_{\langle j_m, k_m \rangle}(p)(0)), \delta_Y G_{\langle j_m, k_m \rangle}(p)) \\
 &< 2^{-m} + 2^{-m} + 2^{-k_m-3} + 2^{-k_m} \\
 &< 2^{-m+2}
 \end{aligned}$$

for all $m \in \mathbb{N}$. Hence,

$$\lim_{m \rightarrow \infty} \delta_Y G_{\langle j_m, k_m \rangle}(p) = \lim_{m \rightarrow \infty} y'_m = y,$$

$$\text{i.e. } y \in \overline{\bigcup_{\langle i, k \rangle=0}^{\infty} \{\delta_Y G_{\langle i, k \rangle}(p)\}}. \quad \square$$

Now we want to state a technical lemma which allows to handle product spaces. If (X, d_X, D_X, α_X) , (Y, d_Y, D_Y, α_Y) are densely enumerated metric spaces then $(X \times Y, d_{X \times Y}, D_{X \times Y}, \alpha_{X \times Y})$ with

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) := \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

and

$$\alpha_{X \times Y} \langle n, k \rangle := (\alpha_X(n), \alpha_Y(k))$$

for all $x_1, x_2 \in X, y_1, y_2 \in Y, n, k \in \mathbb{N}$ is a densely enumerated metric space which is (semi-)computable if X, Y are (semi-)computable (cf. [37, Example 2.3(7)]).

Lemma 32 (Product representation). *Let (X, d_X, D_X, α_X) , (Y, d_Y, D_Y, α_Y) be densely enumerated metric spaces with computably admissible representations δ_X, δ_Y . Let $\delta_{X \times Y}$ be a computably admissible representation of $(X \times Y, d_{X \times Y}, D_{X \times Y}, \alpha_{X \times Y})$. Then*

$$[\delta_X, \delta_Y] \equiv_C \delta_{X \times Y}.$$

A corresponding statement holds for δ_X and $\text{id}_{\mathbb{N}}$. The proof is left to the reader. The previous lemma states that we need not to distinguish computability w.r.t. $\delta_{X \times Y}$ from computability w.r.t. $[\delta_X, \delta_Y]$. Hence we can formulate the following corollary.

Corollary 33 (Computability of recursive relations). *Let (X, d_X, D_X, α_X) , (Y, d_Y, D_Y, α_Y) be semi-computable complete metric spaces. Furthermore, let $(X, Y, \mathbb{R}, \mathbb{N}, \rho)$ be a recursive space system induced by a base β such that all relations in β are densely computable w.r.t. $\text{id}_{\mathbb{N}}$ and computably admissible representations $\delta_X, \delta_Y, \delta_{\mathbb{R}}$. Then for each relation $R : \subseteq X \leftrightarrow Y$ the following holds:*

$$R \in \rho \implies R \text{ densely } (\delta_X, \delta_Y)\text{-computable.}$$

7. Densely computable relations are recursive

In this section we prove that densely computable relations in computable metric spaces are recursive w.r.t. a suitable recursive space system. Actually, we prove a stronger result which will be stated in the following corollary as a normal form theorem.

The main idea of the proof is to use the relaxed representation δ_X . The compact fibers of this representation allow to parallelly perform a computation on all names of an input. Thereby prefix sets of names are determined with the help of the relaxed order which is very closely related to the relaxed representation.

Theorem 34 (Recursiveness of densely computable relations). *Let (X, d_X, D_X, α_X) , (Y, d_Y, D_Y, α_Y) be densely enumerated complete metric spaces with computably admissible representations δ_X, δ_Y and let X be computable. Furthermore, let $(X, Y, \mathbb{R}, \mathbb{N}, \rho)$ be a recursive space system such that*

- (1) $\beta_{\mathbb{R}} \cup \beta_{\mathbb{N}} \cup \pi(X, Y, \mathbb{R}, \mathbb{N}) \subseteq \rho$,
- (2) $d_X : X \times X \rightarrow \mathbb{R}, \alpha_X : \mathbb{N} \rightarrow X, \alpha_Y : \mathbb{N} \rightarrow Y \in \rho$.

Then for $R : \subseteq X \leftrightarrow Y$ the following holds:

$$R \text{ strongly densely } (\delta_X, \delta_Y)\text{-computable} \implies R \in \rho.$$

Proof. By Lemma 26 and 30 it suffices to prove the statement for the relaxed representation $\delta_X : \subseteq \mathbb{B} \rightarrow X$ of X and the Cauchy representation $\delta_Y : \subseteq \mathbb{B} \rightarrow Y$ of Y . Let $R : \subseteq X \leftrightarrow Y$ be strongly densely (δ_X, δ_Y) -computable via a computable function $F : \subseteq \mathbb{B} \rightarrow \mathbb{B}$, i.e.

$$(*) \quad \overline{\bigcup_{i=0}^{\infty} \{\delta_Y F_i(p)\}} = R\delta_X(p)$$

for all $p \in \text{dom}(R\delta_X)$ and $p \notin \text{dom}(F)$ for all $p \in \text{dom}(\delta_X) \setminus \text{dom}(R\delta_X)$. Since F is computable there is a computable function $\varphi : \mathbb{N}^* \times \mathbb{N} \rightarrow \mathbb{N}^*$ which is isotone in the first argument (cf. [37, 2.3.12(3)]), i.e.¹⁰

$$v \sqsubseteq w \implies \varphi(v, n) \sqsubseteq \varphi(w, n)$$

for all $v, w \in \mathbb{N}^*, n \in \mathbb{N}$ and

$$F_i(p) = \begin{cases} \sup_{w \sqsubseteq p} \varphi(w, i) & \text{if the length of the supremum is not finite,} \\ \text{div} & \text{else} \end{cases}$$

for all $p \in \mathbb{B}$. Hence there is a computable function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\overline{\varphi(w, i)} = f(\overline{w}, i)$$

¹⁰ We use the notation $v \sqsubseteq w : \iff (\exists u \in \mathbb{N}^*)vu = w$ for the prefix relation.

for all $w \in \mathbb{N}^*$, $i \in \mathbb{N}$, where $\overline{} : \mathbb{N}^* \rightarrow \mathbb{N}$ is a bijective standard notation and $v^* : \mathbb{N} \rightarrow \mathbb{N}^*$ is the inverse numbering. Now we define some relations:¹¹

- (1) $\text{Guess} : \mathbb{N} \leftrightarrow \mathbb{N}$ by $\text{Guess}(n) := \sum_{i=1}^n \text{Ord}_{\mathbb{R}}(-1/2) = \{0, \dots, n\}$,
- (2) $\text{Bit} : X \times \mathbb{N} \leftrightarrow \mathbb{N}$ by $\text{Bit}(x, \langle n, k \rangle) := (d_X(x, \alpha_X(n)) <_{k+1} 2^{-k})$,
- (3) $\Delta : X \times \mathbb{N} \leftrightarrow \mathbb{N}$ by

$$\Delta(x, k) := \overline{\text{Bit}(x, 0) \cdots \text{Bit}(x, k-1)}$$

$$= \{\bar{w} \in \mathbb{N} \mid \text{lg}(w) = k \text{ and } (\exists p \in \delta_X^{-1}(x)) w \sqsubseteq p\},$$

- (4) $t : \mathbb{N} \times \mathbb{N} \rightarrow Y \times \mathbb{N}$ by $t(n, \bar{w}) := (\alpha_Y \pi(\bar{w}), (n+1) \dot{-} \text{lg}(w))$,
- (5) $T : (X \times \mathbb{N}) \times \mathbb{N} \leftrightarrow Y \times \mathbb{N}$ by $T((x, n), k) := t(n, f(\Delta(x, k) \times \text{Guess}(n)))$,
- (6) $P : \subseteq X \times \mathbb{N} \leftrightarrow Y$ by $P(x, n) := \mu T(x, n)$,
- (7) $Q : \subseteq X \leftrightarrow Y$ by $Q(x) := \text{Lim } P(x)$

for all $x \in X, n, k, \bar{w} \in \mathbb{N}$, where $\pi : \mathbb{N} \rightarrow \mathbb{N}$, $\overline{a_0 \cdots a_{k-1}} \mapsto a_{k-1}$ is a special projection.

We observe that

- (1) $\text{Guess}, \text{Bit}, \Delta, t, T, P, Q \in \rho$,
- (2) $\text{Guess}, \text{Bit}, \Delta, t, T, P$ have finite images,
- (3) $\text{Guess}, \text{Bit}, \Delta, t, T$ are total.

Claim 1.

$$(\forall x \in X)(x \in \text{dom}(R)) \iff (\forall n)(x, n) \in \text{dom}(P)).$$

Proof. Let $x \in X$. Since δ_X has compact fibers, $\delta_X^{-1}(x)$ is compact. Assume there is a $n \in \mathbb{N}$ such that

$$M := \{\bar{w} \in \Delta(x, k) \mid k \in \mathbb{N} \text{ and } (\exists i \leq n) \text{lg } \varphi(w, i) < n+1\}$$

is infinite. Then König's Lemma (cf. [26, 9.7, p. 133]) yields a sequence $(w_j)_{j \in \mathbb{N}}$ in M and an $i \leq n$ such that $p := \sup_{j \in \mathbb{N}} w_j \in \delta_X^{-1}(x)$ and $\text{lg } \varphi(w_j, i) < n+1$ for all $j \in \mathbb{N}$, i.e. $p \notin \text{dom}(F_i) = \text{dom}(F)$. We conclude,

$$\begin{aligned} x \in \text{dom}(R) &\iff \delta_X^{-1}(x) \subseteq \text{dom}(F) \\ &\iff (\forall n)(\exists k)(\forall \bar{w} \in \Delta(x, k))(\forall i \leq n) \text{lg } \varphi(w, i) \geq n+1 \\ &\iff (\forall n)(\exists k) T((x, n), k) = t(n, f(\Delta(x, k) \times \text{Guess}(n))) \subseteq Y \times \{0\} \\ &\iff (\forall n)(x, n) \in \text{dom}(P) = \text{dom}(\mu T), \end{aligned}$$

since T is total. \square

Claim 2.

$$(\forall x \in \text{dom}(R))(\forall m > n) d_Y^<(P(x, n), P(x, m)) < 2^{-n}.$$

¹¹ We use the notation $\text{lg} : \mathbb{N}^* \rightarrow \mathbb{N}$, $a_1 \cdots a_n \mapsto n$ for the *length function* and $\dot{-}$ for the *arithmetic difference* where $n \dot{-} k = 0 : \iff n \leq k$ for all $n, k \in \mathbb{N}$.

Proof. Let $x \in \text{dom}(R)$ and $m > n$. By Claim 1 $(x, n), (x, m) \in \text{dom}(P)$. Let $y \in P(x, n)$. Then there are $k \in \mathbb{N}, i \leq n$ and there is a $\bar{w} \in \Delta(x, k)$ such that

$$y = \alpha_Y \pi f(\bar{w}, i) \quad \text{and} \quad n' := \lg \varphi(w, i) \geq n + 1.$$

Therefore, there is a $p \in \delta_X^{-1}(x)$ such that $w \sqsubseteq p$. Since $(x, m) \in \text{dom}(P)$ there is a $k' \in \mathbb{N}$ and a $\bar{v} \in \Delta(x, k')$ with $w \sqsubseteq v \sqsubseteq p$ such that

$$z := \alpha_Y \pi f(\bar{v}, i) \in P(x, m) \quad \text{and} \quad m' := \lg \varphi(v, i) \geq m + 1.$$

Therefore, $m' \geq n'$ and

$$\begin{aligned} d_Y(y, z) &= d_Y(\alpha_Y \pi f(\bar{w}, i), \alpha_Y \pi f(\bar{v}, i)) \\ &= d_Y(\alpha_Y(F_i(p)(n' - 1)), \alpha_Y(F_i(p)(m' - 1))) \\ &< 2^{-n'+1} \leq 2^{-n}. \end{aligned}$$

Since $P(x, n)$ is finite

$$d_Y^<(P(x, n), P(x, m)) = \max_{y \in P(x, n)} d_Y(y, P(x, m)) < 2^{-n}$$

follows. \square

Claim 3. $Q = R$.

Proof. By Claims 1 and 2 we have $\text{dom}(Q) = \text{dom}(\text{Lim } P) = \text{dom}(R)$. Let $x \in \text{dom}(R)$. We have to show $R(x) = \text{Lim } P(x)$.

“ \subseteq ” Let $y \in R(x)$. By Lemma 9 it suffices to show $y \in \lim_{n \rightarrow \infty}^> P(x, n) = \text{Lim } P(x)$, i.e.

$$\liminf_{n \rightarrow \infty} d_Y(y, P(x, n)) = 0.$$

Therefore, let $n \in \mathbb{N}$ and $p \in \delta_X^{-1}(x)$. By (*) there is a $i_n \in \mathbb{N}$ such that $d_Y(y, \delta_Y F_{i_n}(p)) < 2^{-n}$. Let¹²

$$k := \min\{k' \in \mathbb{N} \mid \lg \varphi(p[k'], i_n) \geq n + i_n + 1\}.$$

Then $l := \lg \varphi(p[k], i_n) - 1 \geq n + i_n$ and

$$y_n := \alpha_Y(F_{i_n}(p)(l)) = \alpha_Y \pi f(\overline{p[k]}, i_n) \in P(x, n + i_n).$$

Hence,

$$\begin{aligned} d_Y(y, y_n) &\leq d_Y(y, \delta_Y F_{i_n}(p)) + d_Y(\delta_Y F_{i_n}(p), \alpha_Y(F_{i_n}(p)(l))) \\ &< 2^{-n} + 2^{-l} \leq 2^{-n+1}, \end{aligned}$$

i.e. $\lim_{n \rightarrow \infty} d_Y(y, y_n) = 0$ and hence $\lim_{n \rightarrow \infty} d_Y(y, P(x, n + i_n)) = 0$.

¹² We use the notation $p[k] := p(0) \cdots p(k-1) \in \mathbb{N}^*$ for the prefix of $p \in \mathbb{B}$ with length $k \in \mathbb{N}$.

“ \supseteq ” Since $R(x)$ is closed it suffices to show

$$\lim_{n \rightarrow \infty} d_Y^<(P(x, n), R(x)) = 0.$$

Then by Lemmas 8 and 9 $\text{Lim } P(x) = \lim_{n \rightarrow \infty}^> P(x, n) \subseteq R(x)$ follows. Let $n \in \mathbb{N}$ and $y \in P(x, n)$. Then there are $k \in \mathbb{N}, i \leq n$ and a $\bar{w} \in \Delta(x, k)$ such that

$$y = \alpha_Y \pi f(\bar{w}, i) \text{ and } n' := \lg \varphi(w, i) \geq n + 1.$$

Therefore there is a $p \in \delta_X^{-1}(x)$ such that $w \sqsubseteq p$ and by (*)

$$d_Y(y, R(x)) \leq d_Y(\alpha_Y(F_i(p)(n' - 1)), \delta_Y F_i(p)) \leq 2^{-n}$$

and

$$d_Y^<(P(x, n), R(x)) = \sup_{y \in P(x, n)} d_Y(y, R(x)) \leq 2^{-n}. \quad \square$$

It is easy to deduce a normal form theorem from the previous theorem which corresponds to Kleene’s Normal Form Theorem.

Corollary 35 (Normal form). *Let (X, d_X, D_X, α_X) , (Y, d_Y, D_Y, α_Y) be densely enumerated complete metric spaces with computably admissible representations δ_X, δ_Y and let X be computable. Furthermore, let $(X, Y, \mathbb{R}, \mathbb{N}, \rho)$ be the recursive space system induced by the base $\beta := \beta_{\mathbb{R}} \cup \beta_{\mathbb{N}} \cup \{d_X, \alpha_X, \alpha_Y\}$. Then for each strongly densely (δ_X, δ_Y) -computable relation $R : \subseteq X \leftrightarrow Y$ there is a total relation $T : (X \times \mathbb{N}) \times \mathbb{N} \leftrightarrow Y \times \mathbb{N}$ with finite images which is algebraically recursive w.r.t. β such that*

$$R = \text{Lim } \mu T.$$

8. Recursive and densely computable relations in analysis

Now we want to apply the results of the previous subsections to the spaces \mathbb{R} and $\mathcal{C}[0, 1]$. Consider the computable metric space $(\mathbb{R}, d_{\mathbb{R}}, \mathbb{Q}, v_{\mathbb{Q}})$ with the metric $d_{\mathbb{R}}$ defined by $d_{\mathbb{R}}(x, y) := |x - y| = \sqrt{(x - y)^2}$ for all $x, y \in \mathbb{R}$, the numbering $v_{\mathbb{Q}} : \mathbb{N} \rightarrow \mathbb{Q}$, $\langle n, k, m \rangle \mapsto (n - k)/(m + 1)$, and the corresponding Cauchy representation $\delta_{\mathbb{R}}$. Furthermore let $(\mathbb{R}, \mathbb{N}, \rho_{\mathbb{R}})$ be the recursive metric space with the base $\beta := \beta_{\mathbb{R}} \cup \beta_{\mathbb{N}}$, as introduced in Definition 18. Obviously $d_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $v_{\mathbb{Q}} : \mathbb{N} \rightarrow \mathbb{R}$ are in $\rho_{\mathbb{R}}$. It is well-known that the functions in β are computable w.r.t. $\text{id}_{\mathbb{N}}, \delta_{\mathbb{R}}$. For completeness we prove:

Lemma 36 (Computability of the relaxed order). *$\text{Ord}_{\mathbb{R}}$ is densely $(\delta_{\mathbb{R}}, \text{id}_{\mathbb{N}})$ -computable.*

Proof. It is well known that the sets $A := \{x \in \mathbb{R} \mid x < 0\}$ and $B = \{x \in \mathbb{R} \mid x + 1 > 0\}$ are recursively enumerable (cf. [37]). Hence there are computable functions

$F, G : \mathbb{B} \times \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\delta_{\mathbb{R}}(p) \in A \iff (\exists n)F(p, n) = 0 \quad \text{and} \quad \delta_{\mathbb{R}}(p) \in B \iff (\exists n)G(p, n) = 0$$

for all $p \in \text{dom}(\delta_{\mathbb{R}})$. Define $H : \mathbb{B} \rightarrow \mathbb{B}$ by

$$H_i(p) := \begin{cases} 0 & \text{if } i = 2n \text{ and } F(p, n) = 0, \\ 1 & \text{if } i = 2n + 1 \text{ and } G(p, n) = 0, \\ k & \text{else,} \end{cases}$$

where $m := \min\{n \in \mathbb{N} \mid F(p, n) = 0 \text{ or } G(p, n) = 0\}$ and

$$k := \begin{cases} 0 & \text{if } F(p, m) = 0, \\ 1 & \text{else} \end{cases}$$

for all $p \in \mathbb{B}, i \in \mathbb{N}$. Therefore, H is computable and $\text{Ord}_{\mathbb{R}}$ is densely $(\delta_{\mathbb{R}}, \text{id}_{\mathbb{N}})$ -computable via H , i.e.

$$\overline{\bigcup_{i=0}^{\infty} \{H_i(p)\}} = \text{Ord}_{\mathbb{R}} \delta_{\mathbb{R}}(p)$$

for all $p \in \text{dom}(\delta_{\mathbb{R}})$. \square

As a direct consequence of Corollary 33 and Theorem 34 we get

Corollary 37 (Relations in \mathbb{R}). *Let $R : \subseteq \mathbb{R} \leftrightarrow \mathbb{R}$ be a relation. Then*

- (1) $R \in \rho_{\mathbb{R}} \implies R$ densely $(\delta_{\mathbb{R}}, \delta_{\mathbb{R}})$ -computable,
- (2) R strongly densely $(\delta_{\mathbb{R}}, \delta_{\mathbb{R}})$ -computable $\implies R \in \rho_{\mathbb{R}}$.

An analogous statement holds for functions $f : \subseteq \mathbb{R} \rightarrow \mathbb{R}$.

Now we consider the computable metric space $(\mathcal{C}[0, 1], d_{\mathcal{C}[0, 1]}, \mathbb{Q}[x], v_{\mathbb{Q}[x]})$ where $d_{\mathcal{C}[0, 1]}$ with $d_{\mathcal{C}[0, 1]}(f, g) := \|f - g\|$ for all $f, g \in \mathcal{C}[0, 1]$ is the usual supremum metric, where $\mathbb{Q}[x]$ is the set of all polynomials with rational coefficients, and where $v_{\mathbb{Q}[x]} : \mathbb{N} \rightarrow \mathbb{Q}[x]$ is the numbering, defined by

$$v_{\mathbb{Q}[x]}(m) = f : \iff v^*(m) = k_0 \cdots k_n \in \mathbb{N}^* \quad \text{and} \quad f = \sum_{i=0}^n v_{\mathbb{Q}}(k_i)x^i$$

for all $n, m \in \mathbb{N}$ and $f \in \mathbb{Q}[x]$. Let $\delta_{\mathcal{C}[0, 1]}$ be the corresponding Cauchy representation. Furthermore, let $(\mathcal{C}[0, 1], \mathbb{R}, \mathbb{N}, \rho_{\mathcal{C}[0, 1]})$ be the recursive metric space system with base β , as introduced in Definition 20. Obviously, $d_{\mathcal{C}[0, 1]} : \mathcal{C}[0, 1] \times \mathcal{C}[0, 1] \rightarrow \mathbb{R}$ and $v_{\mathbb{Q}[x]} : \mathbb{N} \rightarrow \mathcal{C}[0, 1]$ are in $\rho_{\mathcal{C}[0, 1]}$.

As a direct consequence of Corollary 33 and Theorem 34, we get

Corollary 38 (Relations in the space of continuous functions). *Let $R : \subseteq \mathcal{C}[0, 1] \leftrightarrow \mathcal{C}[0, 1]$ be a relation. Then*

- (1) $R \in \rho_{\mathcal{C}[0, 1]} \implies R$ densely $(\delta_{\mathcal{C}[0, 1]}, \delta_{\mathcal{C}[0, 1]})$ -computable,

(2) R strongly densely $(\delta_{\mathcal{C}[0,1]}, \delta_{\mathcal{C}[0,1]})$ -computable $\implies R \in \rho_{\mathcal{C}[0,1]}$.
 An analogous statement holds for functions $f: \subseteq \mathcal{C}[0,1] \rightarrow \mathcal{C}[0,1]$.

9. Conclusion

In this paper we have presented a recursive characterization of computable operations on the real numbers and on other metric spaces. We have not only characterized extensional operations but intensional operations, which appear as relations, too. Indeed this approach is sufficiently rich to cover all intensional operations: a selection theorem, which states that each computable relation has a densely computable subrelation (with the same domain) and with compact images, can be found in [3] (a second characterization of densely computable relations is included too).

Of course, one can imagine several improvements and variants of the presented characterization. It would be a little bit farther away from the classical approach, but maybe more natural, to take the projections as operators and not as basic functions (in this case an additional product operator $f \times g$ and the identities as basic functions should be added) and to replace the limit operator by a limit basic function (which requires sequence spaces X^ω as basic sets and corresponding variants of the operators). This modification would lead to a clear distinction between basic operations and operators: the basic functions would represent the structure of the considered spaces and the operators would be simple set theoretic ones, which do not refer to this structure (provided that the closure in the composition and iteration operator would be eliminated).

A lot of questions have been left for further investigations:

(1) How can effectivity properties of sets be characterized in this approach? Especially, a topological classification of the domains of recursive operations in the Borel hierarchy would be interesting.

(2) Is there a corresponding recursive characterization of computable functions without relations? Especially: which class of functions can be generated if one ommits the relaxed order relation?

(3) Can this approach be extended to a reasonable generalized recursion theory?

Furthermore, there are some questions concerning related and more practical subjects. One can introduce a real random access machine model, based on the basic functions presented in this paper. Peter Hertling and the author have shown (cf. [5]) that such a machine model with a logarithmic time complexity measure is polynomially realistic in comparison with Turing machines in the sense of Ko (cf. [20]). Last not least a corresponding programming language could be introduced and investigated.

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